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Lattice solitary wave solutions of discrete nonlinear wave equations using a direct method

Yi Xiao† and Wen-hau Hai‡

† Department of Physics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China

‡ Department of Physics, Hunan Education Institute, Changsha 410012, People's Republic of China

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Abstract. It is shown that the real exponential approach to the solutions of nonlinear evolution and wave equations can also be applied to discrete systems. As examples, the lattice solitary wave and kink solutions of the discrete modified KdV (DMKdV) equation and some modified versions of this equation are derived using this approach

1. Introduction

There are several standard methods of obtaining solitary wave solutions of nonlinear evolution and wave equations. Among them are direct integration (whenever possible), the inverse scattering method [1], the Backlund transformation [2, 3], the Darboux transformation [4, 5], the Hirota method [6], and the Fredholm determinant method [7]. Recently a real exponential approach to find the solitary wave solutions of nonlinear evolution and wave equations has been proposed by Korpel [8] and developed by Hereman *et al* [9]. Like the Hirota method, the real exponential approach is a direct method. Since the latter does not need to guess a dependent variable transformation, it is convenient to use it to derive solitary wave solutions of both integrable and non-integrable nonlinear evolution and wave equations. In fact, the real exponential approach has been employed to derive single solitary wave solutions of a large class of nonlinear evolution and wave equations successfully. A comprehensive list of these equations and solutions may be found in [9].

In the real exponential approach, the solution of a nonlinear equation is represented as a series in the real exponential solution of the linearized equation. The coefficients of the series satisfy a highly nonlinear recursion relation. If the nonlinear equation has a solitary wave solution, the series can be summed to arrive at a closed form. Up to now the real exponential approach has been applied only to continuous nonlinear evolution and wave equations. In the present paper, we show that the real exponential approach can also be applied to discrete evolution and wave equations. We shall show this by considering the DMKdV equation and some modified versions of this equation.

In section 2, the solitary wave and kink solutions of the DMKdV equation are derived using the real exponential approach. In section 3, the real exponential approach is generalized to find the solitary wave solutions of a d-dimensional version of the DMKdV equation. In section 4, a modified DMKdV equation is studied. Section 5 contains the conclusions.

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2. The DMKdV equation

The DMKdV equation has the form [10]

$$\dot{\phi}_j = (\phi_{j+1} - \phi_{j-1})(1 + \phi_j^2) \qquad j = \ldots, -2, -1, 0, 1, 2, \ldots \tag{1}$$

where the dot denotes the derivative with respect to time. In order to use the real exponential approach to derive the lattice solitary wave solutions of the DMKdV equation, we expand ϕ_j as a series of the form [9]

$$\phi_j = \sum_{n=1}^{\infty} a_n g_j^n \qquad g_j < 1 \tag{2a}$$

$$\phi_j = \sum_{n=1}^{\infty} a'_n g_j^{-n} \qquad g_j > 1 \tag{2b}$$

where a_n and a'_n are the expansion coefficients and

$$g_j = \exp[-K(j - j_0) + \omega t)] \qquad K > 0 \tag{3}$$

with j_0 being an integer and ω a real quantity. For fixed time t, we can determine a value (i_0) of j from $K(j - j_0) - \omega t = 0$. i_0 may not be an integer. We expand ϕ_j by using (2*a*) when $j > i_0$ and (2*b*) when $j < i_0$. If equations (2*a*) and (2*b*) have the same closed form, then we arrive at a solution of equation (1).

Substituting (2*a*) into (1) and equating the coefficients of g_j^n, we obtain the nonlinear recursion relation for the coefficient a_n,

$$[\omega + 2\sinh(K)]a_1 = 0 \tag{4a}$$

$$[2\omega + 2\sinh(2K)]a_2 = 0 \tag{4b}$$

$$[n\omega + 2\sinh(nK)]a_n = -\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} 2\sinh(1K)a_1 a_{m-1} a_{n-m} \qquad (n \geqslant 3) \tag{4c}$$

Case (i):

$$a_1 = \text{arbitrary constant} \qquad a_2 = 0. \tag{5}$$

Under this condition, equation (4*a*) leads to

$$\omega + 2\sinh(K) = 0 \qquad \text{or } v = \omega/K = -2\sinh(K)/K \tag{6}$$

where v is the velocity of the lattice solitary wave. Then (4) has the non-trivial solution

$$a_{2p} = 0 \qquad p = 1, 2, 3, \ldots \tag{7a}$$

$$a_{2p+1} = \pm(-)^p a_1^{2p+1}/[2\sinh(K)]^{2p} \qquad p = 0, 1, 2, 3, \ldots. \tag{7b}$$

Equation (7*b*) can be proved as follows. First, it is easy to show that (7*a*) holds. Then, using (7*a*) and assuming

$$a_{2p+1} = c(-)^p a^{2p+1} \qquad p = 0, 1, 2, \ldots \tag{8}$$

equation (4c) becomes

$$c^2 = 4\sinh^2(K) \tag{9}$$

where we have used (A5) and (A6) from the appendix. It is clear that (9) holds when $c = \pm 2\sinh(K)$. Let $a = a_1/2\sinh(K)$ and then (8) is identical to (7b).

Thus we obtain

$$\phi_j = \pm 2\sinh(K)[g_j/(1+g_j^2)] \qquad g_j < 1 \tag{10}$$

where we have set $a_1 = \pm 2\sinh(K)$ and used the identity [11]

$$\sum_{n=0}^{\infty}(-)^n x^{2n+1} = x/[1+x^2] \qquad |x| < 1. \tag{11}$$

In a similar way we can obtain

$$\phi_j = \pm 2\sinh(K)[g_j/(1+g_j^2)] \qquad g_j > 1 \tag{12}$$

It is clear that (10) and (12) have the same form. Thus we finally obtain the solitary wave solution of (1)

$$\phi_j = \pm\sinh(K)\,\mathrm{sech}[K(j-j_0)-\omega t)] \tag{13}$$

with

$$\omega = Kv = -2\sinh(K). \tag{14}$$

It is noted that the DMKdV equation is integrable and has N-soliton solutions [10]. Although the real exponential approach can also be used to find the N-soliton solutions [9], its main advantage is to derive the single solitary wave solutions.

Case (ii): $a_1 = 0$ and $a_2 =$ arbitrary constant

It can be shown that in this case the solution ϕ_j of (1) has the same form as (13) with (14) except that K is replaced by $2K$ and ω by 2ω.

Next, we consider another form of the DMKdV equation

$$\dot{\phi}_j = (\phi_{j+1} - \phi_{j-1})(1-\phi_j^2). \tag{15}$$

Since this equation is related to (1) by the simple transformation $\phi_j \to (-1)^{1/2}\phi_j$, real solutions of (15) with vanishing boundary conditions can be found by considering the imaginary solution of (1) with vanishing boundary conditions. For example the cosech solution to (15) can be obtained from the sech solution to (1) by adding a phase shift of $i\pi/2$.

However our interest here is in finding the solitary wave solutions with non-vanishing boundary conditions. It is known that the continuous limit of (15) has solutions with non-vanishing boundary conditions [13]. To look for this kind of solution to (15), we expand ϕ_j by

$$\phi_j = a_0 + \sum_{n=1}^{\infty} a_n g_j^n \qquad g_j < 1 \tag{16a}$$

$$\phi_j = a_0 + \sum_{n=1}^{\infty} a_n g_j^{-n} \qquad g_j > 1 \tag{16b}$$

Substituting (16*a*) into (15) and equating the coefficients of g_j^n, we obtain the recursion relation for a_n,

$$a_1 = \text{arbitrary constant} \tag{17a}$$

$$a_2 = a_0 a_1^2/[(1-a_0^2)2\sinh^2(K/2)] \tag{17b}$$

$$(1-a_0^2)[\sinh(nK) - n\sinh(K)]a_n = 2a_0\sum_{m=1}^{n-1}\sinh(mK)a_m a_{n-m}$$

$$+\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\sinh(1K)a_l a_{m-1}a_{n-m} \qquad (n \geqslant 3) \tag{17c}$$

where we have taken

$$\omega = -2(1-a_0^2)\sinh(K). \tag{18}$$

The solution a_n of (17) can be written as

$$a_n = \pm a_1^n/[2\tanh(K/2)]^{n-1} \qquad (n = 1, 2, 3, \ldots). \tag{19}$$

This can be proved as follows. For $n = 2$, equation (19) is the same as (17*b*) with $a_0 = \pm\tanh(K/2)$. For $n \geqslant 3$, assuming $a_n = ca^n$, equation (17*c*) becomes

$$(1-a_0^2)[\sinh(nK) - n\sinh(K)] = (2a_0c)\sum_{m=1}^{n-1}\sinh(mK) + c^2\sum_{m=1}^{n-1}\sum_{l=1}^{m-1}\sinh(1K) \qquad (n \geqslant 3). \tag{20}$$

Taking $c = 2a_0$ and using the identity (A3) from the appendix, we obtain

$$(1-a_0^2) = -2a_0^2/[1-\cosh(K)]. \tag{21}$$

It is easy to show that (21) holds for $n \geqslant 3$ when $a_0 = \pm\tanh(K/2)$. Substituting (21) into (16*a*), we find

$$\phi_j = \pm\tanh(K/2)\tanh[K(j - j_0 - vt)/2] \qquad (g_j < 1) \tag{22a}$$

or

$$\phi_j = \pm\tanh(K/2)\coth[K(j - j_0 - vt)/2] \qquad (g_j < 1). \tag{22b}$$

In a similar way, we obtain for $g_j > 1$

$$\phi_j = \pm\tanh(K/2)\tanh[K(j - j_0 - vt)/2] \qquad (g_j > 1) \tag{23a}$$

or

$$\phi_j = \pm\tanh(K/2)\coth[K(j - j_0 - vt)/2] \qquad (g_j > 1). \tag{23b}$$

From equations (22) and (23), we find the following possible solutions to (15)

$$\phi_j = \pm\tanh(K/2)\tanh[K(j - j_0 - vt)/2] \tag{24}$$

$$\phi_j = \pm\tanh(K/2)\coth[K(j - j_0 - vt)/2] \tag{25}$$

with

$$v = -2\sinh(K)\tanh(K)/K. \tag{26}$$

The solution (24) is a kink solution and (25) is a singular solution of equation (15).

3. A *d*-dimensional DMKdV equation

The real exponential approach can be generalized to derive the solitary wave solutions of some d-dimensional discrete nonlinear evolution and wave equations. As an example, we consider a d-dimensional version of the DMKdV equation [12]

$$\dot{\phi}(\boldsymbol{j}) = \sum_{\alpha=1}^{d} B_\alpha[\phi(\boldsymbol{j}+\boldsymbol{e}_\alpha) - \phi(\boldsymbol{j}-\boldsymbol{e}_\alpha)][1+\phi(\boldsymbol{j})^2] \tag{27}$$

where $\boldsymbol{j} = (j_1, j_2, \ldots, j_d)$ is a vector with its αth element being an integer j_α and $\boldsymbol{e}_\alpha$ is the unit vector in the direction of the αth axis of a simple cubic lattice.

To find the solitary wave solution of (27), we need only to replace g_j in (2) by

$$g(j) = \exp[-(\boldsymbol{K}\cdot(\boldsymbol{j}-\boldsymbol{j}_0) - \omega t)] \qquad \boldsymbol{K} = (K, K, \cdots, K) \qquad K > 0 \tag{28}$$

with $\boldsymbol{j}_0$ being a constant vector. Then it is easy to show that the solitary wave solution of (27) is

$$\phi(\boldsymbol{j}) = \pm\sinh(K)\,\mathrm{sech}[\boldsymbol{K}\cdot(\boldsymbol{j}-\boldsymbol{j}_0) - \omega t)] \tag{29}$$

with

$$\omega = -2B\sinh(K) \qquad B = \sum_{\alpha=1}^{d} B_\alpha. \tag{30}$$

Similarly the solitary wave solutions of the equation

$$\dot{\phi}(\boldsymbol{j}) = \sum_{\alpha=1}^{d} B_\alpha[\phi(\boldsymbol{j}+\boldsymbol{e}_\alpha) - \phi(\boldsymbol{j}-\boldsymbol{e}_\alpha)][1-\phi(\boldsymbol{j})^2] \tag{31}$$

can be found to be

$$\phi(\boldsymbol{j}) = \pm\tanh(K/2)\tanh[(\boldsymbol{K}\cdot(\boldsymbol{j}-\boldsymbol{j}_0) - \omega t)/2] \tag{32}$$

$$\phi(\boldsymbol{j}) = \pm\tanh(K/2)\coth[(\boldsymbol{K}\cdot(\boldsymbol{j}-\boldsymbol{j}_0) - \omega t)/2] \tag{33}$$

with

$$\omega = -2B\sinh(K)\tanh(K). \tag{34}$$

4. A modified DMKdV equation

In this section we consider a modified DMKdV equation and its d-dimensional version. This modified DMKdV equation is written as [12]

$$\dot{\phi}_j = (\phi_{j+1} - \phi_{j-1})(1+\phi_j)^2. \tag{35}$$

Substituting (2*a*) into (35), we obtain the recursion relation

$$[\omega + 2\sinh(K)]a_1 = 0 \tag{36a}$$

$$[2\omega + 2\sinh(2K)]a_2 = -4a_1^2 \sinh(K) \tag{36b}$$

$$[n\omega + 2\sinh(nK)]a_n = -4\sum_{m=1}^{n-1} \sinh(mK) a_m a_{n-m}$$

$$- \sum_{m=2}^{n-1}\sum_{l=1}^{m-1} 2\sinh(1K) a_1 a_{m-1} a_{n-m} \qquad (n \geqslant 3). \tag{36c}$$

The solution of (36) is found to be

$$a_n = (-)^{n-1} n a_1^n / 4\sinh^2(K/2)]^{n-1} \tag{37}$$

with

$$\omega = -2\sinh(K) \qquad a_1 = \text{arbitrary constant.} \tag{38}$$

This can be proved as follows. For $n = 2$ it is shown that (36*b*) is the same as (37). For $n \geqslant 3$, using (38) and taking $a_n = cna^n$, equation (36*c*) becomes

$$[n\sinh(K) - \sinh(nK)]n = 2c\sum_{m=1}^{n-1} m(n-m)\sinh(mK)$$

$$+ c^2 \sum_{m=2}^{n-1}\sum_{l=1}^{m-1} 1(m-1)(n-m)\sinh(1K). \tag{39}$$

Using (A7) and (A8), we can prove the following identity

$$\sum_{m=1}^{n-1} m(n-m)\sinh(mK)$$

$$= -[n(1-\cosh(K))\sinh(nK) + \sinh(K)(\cosh(nK) - 1)]/[2(1-\cosh(k))^2]. \tag{40}$$

Using (40), we find that (39) holds when $c = 2[1 - \cosh(K)] = -4\sinh^2(K/2)$. Let $a = a_1/[-4\sinh^2(K/2)]$ and then we arrive at equation (37).

In a similar way, we can find a_n^l for $g_j > 1$. Then the solitary wave solution of (35) is found to be

$$\phi_j = \sinh^2(C)\,\text{sech}^2[C(j - j_0) - \omega t] \tag{41}$$

with

$$\omega = -\sinh(2C) \tag{42}$$

where $C = K/2$. To ensure (2*a*) and (2*b*) have the same closed form, we have set $a_1 = 4\sinh^2(C)$.

5. Conclusions

In this paper we have applied the real exponential approach to discrete nonlinear wave equations. We have obtained the solitary wave and kink solutions of the DMKdV equation and its modified versions. We have also shown that the real exponential approach can be applied to the d-dimensional version of the DMKdV equation. Compared with continuous cases, the nonlinear recursion relation of discrete cases always include hyperbolic functions and so it is more difficult to find solutions of the expansion coefficients. Fortunately, when the discrete equation has solitary wave solutions of simple shapes (e.g., sech, tanh and so on), these solutions can also be obtained easily. In the following papers we shall apply the real exponential approach to other integrable and non-integrable discrete wave equations.

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Appendix

In this appendix, we derive some identities used in the text. In [11] we find the following identities:

$$\sum_{m=1}^{n-1} p^m \sinh(mx) = [p \sinh(x) - p^n \sinh(nx) + p^{n+1} \sinh(n-1)x]/[1 - 2p\cosh(x) + p^2] \tag{A1}$$

$$\sum_{m=0}^{n-1} p^m \cosh(m) = [1 - p\cosh(x) - p^n \cosh(n) + p^{n+1} \cosh(n-1)x]/[1 - 2p\cosh(x) + p^2]. \tag{A2}$$

When $p = +1$, equations (A1) and (A2) become

$$\sum_{m=1}^{n-1} \sinh(mx) = [\sinh(x) - \sinh(nx) + \sinh(n-1)x]/[2(1 - \cosh(x))] \tag{A3}$$

$$\sum_{m=0}^{n-1} \cosh(mx) = [1 - \cosh(x) - \cosh(nx) + \cosh(n-1)x]/[2(1 - \cosh(x))] \tag{A4}$$

respectively.

Subtracting (A1) with $p = +1$ from (A1) with $p = -1$, we obtain

$$\sum_{s=1}^{q-1} \sinh(2s+1)x = [\cosh(2qx) - 1]/[2\sinh(x)] \tag{A5}$$

and adding (A2) with $p = +1$ and with $p = -1$ leads to

$$\sum_{s=1}^{q} \cosh(2s)x = [\sinh(2q+1)x - \sinh(x)]/[2\sinh(x)]. \tag{A6}$$

Finally, the first differentiation of (A1) with respect to p for $p = +1$ leads to

$$\sum_{m=1}^{n-1} m \sinh(mx) = [n \sinh(n-1)x - (n-1)\sinh(nx)]/[2(1-\cosh(x))] \tag{A7}$$

and the second differentiation of (A1) with respect to p for $p = +1$ leads to

$$\sum_{m=1}^{n-1} m^2 \sinh(mx) = [n \sinh(n-1)x - (n^2 - 2n)\sinh(nx)]/[2(1-\cosh(x))] + \sinh(x)[\cosh(nx) - 1]/[2(1-\cosh(x))^2]. \tag{A8}$$

References

[1] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095

[2] Miura R M (ed) 1976 *Backlund Transformation (Springer Lecture Notes in Mathematics 515)* (New York: Springer)

[3] Rogers C and Shadwick W R 1982 *Backlund Transformation and their Applications* (New York: Academic)

[4] Levi D, Ragnisco O and Sym A 1982 *Lett. Nuovo Cimento* **33** 401

[5] Xiao Y 1989 *Acta Phys. Sin.* **38** 1911

[6] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192

[7] Poppe C 1983 *Physica* D **9** 103

[8] Korpel A 1978 *Phys. Lett.* **68A** 179

[9] Hereman W, Banerjee P P, Korpel A, Assanto G, Immerzeele A, Vanand A and Meerpeel 1986 *J. Phys. A: Math. Gen.* **19** 697

[10] Ablowitz M J 1978 *Stud. Appl. Math.* **58** 17

[11] Gradshteyn I S and Ryzhik I 1980 *Tables of Integrals, Series and Products* (New York: Academic)

[12] Takeno S 1992 *J. Phys. Soc. Japan* **61** 1433

[13] Chen Z Y, Huang N N, Liu Z Z and Xiao Y 1993 *J. Phys. A: Math. Gen.* **26** 1365